Logical Form

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Abstract. The notion of the logical form of an expression E is shown to be a semantic notion that can be derived from the notion of the structured meaning of E. This simple idea can be traced to categorial grammars, and it is implicitly used by Richard Montague. We argue that a most fine-grained tool for defining structured meaning can be found in Pavel Tichý's Transparent Intensional Logic (TIL). Structured meanings of expressions are identified with abstract procedures, known as TIL constructions, expressed by the expressions. We construe concepts as closed constructions, and present a method of seeking *the* best semantic analysis (identical to the structured meaning) of a given expression. The method terminates in a complete lattice over the set of possible analyses; which analysis is the best one must, however, be relativized to a conceptual system. This relativization concerns primitive concepts of the conceptual system within which the semantic analysis is set. Our main thesis is: *Every well-formed expression E of the language under analysis can be associated with a logical form that is unambiguously derived from the structured meaning of E.*

1 The problem of adequate explication

Logic is formal in the sense that all arguments of the same form as logically valid arguments are also logically valid and hence truth-preserving.

(M. Goméz – Torrente)

The goal of logic is to discover the entailment relation wherever it holds. The urgency of this task becomes obvious as soon as logic is applied to analyzing natural language expressions, because the historically developed grammatical structure of a natural language hides the logical form that underlies the given expressions. Our intuition is that the more fine-grained the analysis, the more instances of entailment are discovered. The question whether an analysis is better (because more fine-grained) than another one is thus connected with the task of explicating the notion *logical form*. In this paper we present a method that makes it possible to find *the* most fine-grained analysis of an expression.

The following question should be answered first: Let E be an expression of a natural language. Can we speak about *the* (unambiguously determined) logical form of E?

Peregrin in [23, p. 1] presents an argument against what he calls a "metaphysical" conception of logic, according to which logic spells out a specific kind of mathematical structure that is somehow inherently related to our actual reasoning. In contrast, he argues that it is always an empirical question whether a given mathematical structure really does capture a principle of reasoning. More generally, he argues that it is not meaningful to replace an empirical investigation of a thing by an investigation of its *a priori* analyzable structure without paying due attention to the question of whether it really is the structure of the thing in question.

Indeed, from the viewpoint of general linguistics, the fact that an expression denotes just what it denotes (in the given language) is contingent: Linguistics is an empirical science, so the logical form of an expression is something what must be discovered in the "the realm of the formal" (see [23]); thus, what we call *logical analysis of language*, LANL (see, e.g., [17, 19]) should accept the following principle ([30], [31, pp.801-842]):

> The notion of a code presupposes that prior to, and independently of, the code itself there is a range of items to be encoded in it. Hence... meanings cannot be conceived of as products of the language itself. They must be seen as logical rather than linguistic structures, amenable to investigation quite apart from their verbal embodiments in any particular language. To investigate logical constructions in this way is the task of logic. The linguist's brief is to investigate how logical constructions are encoded in various vernaculars.

So LANL is based on some study of logical structures and their properties and relations, and this activity is devoid of any empirical factor. As soon as the results of the logical investigation are applied (or made available to application) to the search for the logical form of an expression, we must be aware of the hypothetical character of this application. It is linguists who should participate in checking the adequacy of the proposal. All the same, LANL cannot be reduced to making particular proposals. Richard Montague [21] was the first to demonstrate the possibility of building up a logical *system* that makes it possible to systematically put forward logical constructions as logical (as opposed to grammatical) forms of a fragment of English. Here we will outline another systematic tool for the same project as Montague's: Transparent Intensional Logic (TIL). (See, e.g., [29, 31]).

We should, however, mention David Lewis and his [16], which makes two points that are important w.r.t. some trends in 'post-analytic' semantics:

i) Lewis speaks about the "pleasing finitude that prevents Markerese semantics from dealing with the *relations between symbols and the world of non-*

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symbols — that is, with genuinely semantic relations" (emphasis ours) — this goes against the 'horizontalistic tendencies' which reduce semantics to investigating relations between various *expressions*, and

ii) Lewis distinguishes between "the description of possible languages or grammars as abstract semantic systems whereby symbols are associated with aspects of the world" and "the description of the psychological and sociological facts whereby a particular one of these abstract semantic systems is the one used by a person or population", warning that mixing these two topics leads to confusion. So he defends the possibility of carrying out LANL as a non-empirical discipline.

Returning to the problem of explication, we can see that *the* logical form is at most an ideal: The history of logic shows that what could be called *logical form* is dependent on the expressive power of particular logical systems (see [24] for a good illustration of this). Consider the sentence

(1) *Charles is a bachelor, and Peter believes that Charles is married.*

From the viewpoint of truth-functional ('propositional') logic we would say that the logical form of (1) was

(1') $$p \wedge q$$

whereas a combination with Aristotelian logic (predicate logic) would yield

(1") $$\mathrm{P(k)} \wedge \mathrm{Q(p)}$$

while the 'P(k)' alone would receive the Montagovian form

$$\lambda \mathrm{X}\ {}^{\vee}\mathrm{X(k)}({}^{\wedge}\mathrm{B})$$

(X and B ranging, roughly, over properties).

The crucial *criterion* for choosing the right logical form of an expression E is obviously the amount of claims that can be correctly derived from a sentence containing E thanks to the choice. We can illustrate this criterion by means of the sentence (1).

The truth-functional proposal (1') justifies our derivation of the sentences

Charles is a bachelor and *Peter believes that Charles is married.*

The proposal (1") allows us derive, moreover, the sentences

Somebody is a bachelor and *Somebody believes that Charles is married.*

The Montagovian analysis leads, furthermore, to a more specific reformulation of *Charles is a bachelor*:

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Among the properties had by Charles is the property of being a bachelor.

We would like to derive further conclusions, for example,

Charles is not married, or

Peter believes that there are some married individuals.

But not only that. We would also like to *prevent* derivations that are incorrect, i.e., those whose conclusions are *not* entailed by the given sentence(s). There are plenty of such cases in the literature. Here are some of them (see [14, pp. 203, 204]):

The population of Amsterdam equals the population of Rotterdam.
The population of Amsterdam is declining.
⇒ *The population of Rotterdam is declining.*

The treasurer of the charity organization is the chairwoman of the Entertainment Committee.
The treasurer of the charity organization resigns.
⇒ *The chairwoman of the Entertainment Committee resigns.*

Or the classics:

Oedipus seeks the murderer of his father.
Oedipus is the murderer of his father.
⇒ *Oedipus seeks Oedipus.*

If the 1^{st}-order predicate logic (with functional symbols) were used, and if 'is' in the second premise were analyzed as the identity of individuals (as it should be), we would not be able to block the incorrect inference in such cases. We will demonstrate this by means of the last example. The logical forms of the components of the argument would be

$\mathrm{R}(\mathrm{a}, \mathrm{f}(\mathrm{g}(\mathrm{a})))$
$\mathrm{a} = \mathrm{f}(\mathrm{g}(\mathrm{a}))$
$\therefore\ \mathrm{R}(\mathrm{a}, \mathrm{a})$

We cannot say that, e.g., Leibniz' law should be refuted. *We must ask whether the logical forms proposed for the premises and/or conclusion are adequate.* The degree of the adequacy is given by the *criterion* given above.

We should be aware of the fact that various logical forms associated with one and the same expression underlie the expression *objectively*, i.e., independently of our knowledge. The development of natural languages outruns semantic theory, which only gradually discovers the admirable capacities of natural languages. For example, the capacity of Greek to encode the predication relation was there long before Aristotle's discovery of this relation.

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Now, if the candidates for logical form have to be sought in the abstract realm of logical objects and if their evaluation obeys the *criterion* stated above, then a concept of logical form turns out to be a *semantic* concept, however strange this conclusion may seem. Indeed, finding better logical forms means — according to this criterion — finding such logical forms that make it possible to derive more correct inferences. Correct inferences lead exclusively to conclusions that are *entailed*, but entailment is a semantic concept: The conclusion must be *true* in all possible worlds (indices, circumstances etc.) where the premises are *true*. Besides, whether some sentence is true or not, is (co-)determined by the *meaning* of the expressions involved.

The sense of oddity is, however, comprehensible. Logical form is a *form*, and the concept of form is nearly exclusively associated with *syntactic* features. If *meaning* is a set-theoretic entity then we can hardly find something like syntactic features of meaning. Dummett has linked semantic realism with the claim that meaning is truth-conditionally defined, and he does not accept a truth-conditional conception of meaning: His conception of meaning is intuitionistic, and to the extent that intuitionistic philosophy is anti-realistic Dummett is an anti-realist, of course. But Dummett has rightly seen that meanings should be a kind of *procedure*. Intuitionistic procedures ('constructions') are basically mental. Could procedures be so defined that the definition would be acceptable to a realist?

2 Meaning as a procedure. Categorial Grammar. Montague

One of the *syntactic features of meaning* is that meaning should be *structured*, so that the syntactic structure of expressions can somehow mirror the semantic structure, thus making it possible to derive (at least in principle) the meaning of an expression from the syntactic structure of the latter. This requirement was first explicitly formulated by Cresswell [7, 8] but this insight can be tracked back to some earlier logical works of importance.

An ingenious observation in this respect can be found in [3, p. 244], where Bolzano distinguishes between the concept (*Vorstellung an sich*, and therefore also *Begriff*) and its content (*Inhalt*). Whereas the latter is simply a class (*Summe*) of simple subconcepts (hence unstructured), the former consists in some *combination* of the elements of the content.

The next milestone is Carnap's attempt to solve the problems stemming from applying his method of intension and extension to the analysis of attitudinal contexts [4]; his *intensional isomorphism* can be criticized (as it actually was, for instance by Church [5]), but his semantic simulation of syntactic structure should be appreciated as one of the first attempts to draw the attention of logicians to the phenomenon of 'hyperintensionality', as the phenomenon was later dubbed by Cresswell.

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What was later chronologically is, however, neither Montague nor D. Lewis nor Cresswell. In 1968 and 1969 a young Czech logician published two articles: 'Smysl a procedura' ('Sense and Procedure') and 'Intension in Terms of Turing Machines' (see [31, pp. 77-110]). Tichý had realized that semantics should be conducted as a theory of procedures, though this time of *non-mental* procedures. We will see that this view is one of the fundamental meta-principles of TIL.

Also in 1968 Montague writes his 'Pragmatics' (see [21, pp. 95-118]). He does not explicitly proclaim the procedural character of meanings, but his approach to analyzing language shows that he was aware of the problem, as is evident from his choice of the typed λ-calculus as his logical tool. This choice is significant. First, the λ-calculus was invented by Church as an ideal means of handling *functions*. Second, the type-theoretical classification of objects is in good harmony with Ajdukiewicz' idea of *categorial grammar* [1], which was originally meant to analyze syntax, but the background of which was a most natural semantics (syntactic concatenation corresponding to application of a function to its argument). As Arnim v. Stechow points out [27, p. 120] concerning the 'reconstruction' of Ajdukiewicz *via* Montague's Universal Grammar:

1. Es gibt einen strikten Parallelismus zwischen syntaktischen und semantischen Kategorien. ...
2. Als einzige syntaktische Operation ist die Verkettung zugelassen, die so beschränkt ist, dass ein Funktor stets mit dazugehörigen Argumenten verkettet werden darf.
3. Jede solche Syntaxregel ist als funktionale Applikation der Bedeutung des Funktors auf seine Argumente interpretierbar. [1]

Remark. The connection between logical form and the inference relation led Gareth Evans to his definition of *validity in virtue of semantic structure* based on the ideas of categorial grammars. As M. Sainsbury in his [26, p. 308] characterizes Evans' notion, "a structurally valid inference is one whose validity is independent of the particular assignments that are made within the semantic categories, but which is wholly dependent on the pattern of the categories in the argument". Our conception specifies in a way these general characteristics.

Now, it is appropriate to emphasize that starting with such primitive notions as *class, relation* and suchlike differs in an important point from starting with *function* as a primitive. In the latter case the operations known as *application to argument(s)* and λ*-abstraction* become definable: functions are 'procedure-friendly'. [2]

[1] 1. There is a strict parallelism between the syntactic and semantic categories... 2. The only syntactic operation is admitted, namely concatenation, which is constrained as follows: a functor has to be always concatenated with the respective arguments. 3. Any such syntactic rule can be interpreted as the application of the functor's denotation to its arguments.

[2] It is not by chance therefore that mathematicians did not always use the term *function* in

The principles of categorial grammars make a smooth transition from grammatical categories to *types* possible, which is the way chosen by Montague. Montague's system is well-known so a brief summary will suffice. Fundamental grammatical categories (of a fragment of English) are generated in accordance with the definition (see, e.g., [14, p. 151]):

i) S, CN, IV $\in$ CAT

ii) If A, B $\in$ CAT, then A/B $\in$ CAT

The categories sub i) correspond to *sentences, common nouns, intransitive verbs,* respectively; the point ii) means: by applying the category A/B to the category B, we get the category A. Thus, e.g., attitude verbs ("sentential complement verbs") belong to the category IV/S, because by applying them (e.g., *to believe that*) to a sentence (*Prague is a Czech city*) we get an expression that belongs to the category *intransitive verb* (indeed: *to believe that Prague is a Czech city*).

Montague used categorial grammar to translate categorial expressions into an *intensional logic* based on the typed λ-calculus. Montague's atomic types are e ('entity', maybe: individuals) and t (truth-values). Thus, classes of individuals belong to the type $<e, t>$, i.e., they are functions from e to t (characteristic function of the class), while the type of binary relations is $<e, <e, t>>$ (in virtue of Schoenfinkel's reduction of multiple-argument functions to monadic functions)[3].

Montague was, of course, well aware of the fact that natural language needs intensions as objects to talk about. He therefore introduced the 'non-type' s, which is interpreted as the set of possible worlds. (In the 'two-sorted' version of the type theory s is a separate type, see [13].) Thus the type of properties — as opposed to classes — is $<s, <e, t>>$.

Montague tried to handle the *de re* vs. *de dicto* cases by using the symbols $^\wedge$ ('hat', 'cap') and $^\vee$ ('cup'), the former 'activating', the latter 'undoing' the intensional factor. It can be shown that this approach results in a less expressive system than would be desirable. Reinhard Muskens (see [22, p. 10]) showed that the resulting system does not possess the attractive Church-Rosser ('diamond') property. On the other hand, the systems using explicit intenzionalization, i.e., exploiting variables for possible worlds (and time moments) do not have this defect. There are some other reasons to prefer TIL, see, e.g., [29, §28] for critical remarks on Montague's intensional logic.

We will see that some important features are shared by TIL and Montague, first of all the functional approach and an inspiration by the typed λ-calculus.

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its contemporary sense, i.e., as denoting *mappings,* which are set-theoretical objects; functions were previously thought of as *calculations*, see [29]. Also, the original interpretation of terms of lambda calculus was procedural: Barendregt in his [2, p. 184] says: "in this interpretation the notion of a function is taken to be intensional, i.e., as an algorithm."

[3] In the case of partial functions this reduction is not unambiguous so that n-ary function symbols can (and should) be used. See [31, p. 467].

3 Constructions in Transparent Intensional Logic, Ramified Hierarchy of Types

TIL studies abstract objects of a certain kind and the ways they can be constructed. Tichý in [31, p. 295], says:

> Logic is the study of logical objects (individuals, truth-values, possible worlds, propositions, classes, properties, relations, and the like) and of *ways such objects can be constructed from other such objects.* (Emphasis ours.)

As a *logic* TIL investigates the respective objects and constructions independently of the role they may play in LANL. The reason is that

> the nature of such constructions often guarantees noteworthy properties or relationships between the objects generated by those constructions. (*Ibidem.*)

Thus, for example, the way a proposition (as a purely logical object!) is constructed can show that another proposition is entailed by it.

The second reason to examine those objects and constructions is that:

> logical constructions can be assigned to linguistic expressions as their analyses. (*Ibidem.*)

This second reason justifies the role a logic can play in logically analyzing natural language. Indeed, the assignment of constructions to linguistic expressions should answer the old question, *What is the meaning of an expression?* without resorting to a pragmatic "stimulus meaning". (See, e.g., [25].)

For TIL, whose approach is a functional one, the area of objects to be investigated by logic is primarily the set of sets of partial functions definable over the base:

Definition *the base of atomic types of order 1*:

o, i.e., the set of *truth-values* {T, F} (cf. Montague's t),

ι, i.e., the set of *individuals* ('universe') (cf. Montague's e),

τ, i.e., the set of *real numbers / times*,

ω, i.e., the set of *possible worlds.*[4]

***Functional types of order 1*:**

where α, β_1, …, β_m are any *types*, the set of the *partial functions* from $\beta_1 \times \ldots \times \beta_m$ to α, denoted by $(\alpha\beta_1 \ldots \beta_m)$, is a *type.*

[4] A pre-theoretical explication: Let S be a maximum consistent set of (possible) states of affairs. A possible world is a *chronology* of such Ss.

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The assortment of 1st order types is sufficient to meet the requirement that *intensions* should be singled out and distinguished from *extensions.* In TIL it is not sufficient to define extensions as values of intensions: the values of some intensions are themselves intensions (an informal example: consider the expression *the favored proposition of A. Einstein*). So we can define *intensions* as members of the type $((\alpha\tau)\omega)$ (which can be abbreviated as $\alpha_{\tau\omega}$) for any type α.[5] The *non-trivial intensions*, i.e., functions that take distinct values in at least two pairs <possible world, time moment> are of special importance since they can be denotations of *empirical expressions.*

Some examples:

INTENSIONS			EXTENSIONS	
Type	*Name*	*Example*	*Type*	*Example*
$o_{\tau\omega}$	proposition	(that) some mammals live in water	o	truth-value
$\iota_{\tau\omega}$	individual role[6]	the highest mountain	ι	individual
$(o\iota)_{\tau\omega}$	properties of individuals	(being) a dog, to talk	$(o\iota)$	class of individuals
$(o\iota o_{\tau\omega})_{\tau\omega}$	relation between individuals and propositions	believe	$(o\iota\iota)$	linkage (relation-in-extension) between individuals
$\tau_{\tau\omega}$	magnitude	the number of planets	τ	(real) number

Remark: The examples in the third column have to be conceived of as names of extra-linguistic objects. They are English expressions that obviously denote these objects. Yet we should bear in mind that theoretically there are infinitely many English expressions that denote any of those objects: they will be equivalent because of the same *denotation* but not synonymous because they would differ w.r.t. their *meanings* ('senses'). We will soon return to this point.

The inhabitants of the abstract domain defined above are not the only objects to be investigated in logic. An infinite hierarchy of the ways these objects can be *constructed* can be defined so that also these new inhabitants can be constructed, whereby a never-ending hierarchy — the *ramified hierarchy* — is unveiled.

[5] The members of a type $(\alpha\ \omega)$, where $\alpha \neq (\beta\ \tau)$, are intensions too; for example $(o\ \omega)$ is the type of *eternal propositions.*

[6] Tichý uses also the term *individual office*, Church *individual concept.*

First of all *constructions* must be defined. Here we define four of them.[7] The comments following afterwards are meant as a pre-theoretical part of the definition.

***Constructions* — the least set satisfying i) – iv):**

i) *Variables* are *constructions.* They construct objects dependently on valuation, i.e., v-construct, where v is the parameter of valuation.

ii) Let X be any object, even a construction. Then ^{0}X is a *construction* called *trivialization* . It constructs X without any change.

iii) Let X be a *construction* that v-constructs a function F, type $(\alpha\beta_1 \ldots \beta_m)$, and $X_1,\ldots,X_m$ be *constructions* that v-construct respectively $b_1,\ldots,b_m$, objects belonging to the types $\beta_1,\ldots,\beta_m$, respectively. Then $[X\ X_1 \ldots X_m]$ is a *construction* called *composition*; it v-constructs the value of F on $\langle b_1,\ldots,b_m\rangle$. If F is not defined on the respective $\langle b_1,\ldots,b_m\rangle$ then $[X\ X_1 \ldots X_m]$ does not v-construct anything: it is *v-improper.*

iv) Let $x_1,\ldots,x_m$ be pairwise distinct variables ranging respectively over the (not necessarily distinct) types $\beta_1,\ldots,\beta_m$, and let the *construction* X v-construct α-objects. Then $[\lambda x_1 \ldots x_m\ X]$[8] is a *construction* called *closure.* It v-constructs a function F as follows: let v' be any valuation assigning to the variables $x_1,\ldots,x_m$ the objects $b_1,\ldots,b_m$, respectively, and otherwise be identical with v; the value of F on $\langle b_1,\ldots,b_m\rangle$ is the object v'-constructed by X; if X is v'-improper then F is undefined on $\langle b_1,\ldots,b_m\rangle$.

Comments:

Ad i): Informally, constructions are — as abstract procedures — *extra-linguistic* objects. Therefore, variables cannot be letters or characters, as we are used to suppose. They are incomplete constructions. For any type (including the higher-order types) there are countably infinitely many variables at our disposal. For any type α they can be numbered. Members of any type α can be organized in infinitely many sequences. A variable with the index n v-constructs the n-th member of that infinite sequence of members of α which is given by the valuation v. The familiar letters like *'x', 'y', ..., 'p', 'q', ..., 'k', 'l', ..., 'm', 'n'* ... are *names* of variables. Technically, these objectual variables behave just like we know it from Tarski. See [29, §14].

Ad ii): The notion of trivialization was introduced in TIL in [29]. Within the 1st order hierarchy of types the necessity of trivialization was not obvious. Nonetheless, even there we could find some rather epistemological reasons for

[7] In [29] we find six of them. In general, the character of a particular application of TIL and of the problems to be solved determines the choice of constructions. See, e.g., [33] on additional constructions useful in the area of conceptual modelling.

[8] The outmost parentheses can be omitted.

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its introduction: Tichý's pre-1988 conception had it that (1^{st} order) objects were constructions by constructing themselves. So the constructions could contain (1^{st} order) *objects themselves*, which leads to some counterintuitive consequences[9]. Moreover, within the ramified hierarchy such a kind of construction like trivialization is indispensable: *as soon as constructions are to be not only used* to construct objects, *but also mentioned as objects within the system, trivialization needs to be introduced.* A trivialized object is a *mentioned* object.

Ad iii) and iv): These constructions are comprehensible for anybody who knows λ-calculi. One (seemingly slight) distinction between composition and closure, on the one hand, and application and abstraction on the other hand is (at least philosophically) essential: Whereas λ-terms are usually interpreted as *the results* of the respective operation (the application as the value of the respective function on the respective argument, the λ-abstraction as the resulting function), constructions are not artificial expressions which are to be interpreted as results of procedures. Rather, they are the very procedures that lead to the respective results. True, to understand which procedure is relevant we need some means of representing the steps of this procedure, so we define procedures *via* defining — in a language of constructions, if you like — which steps should be carried out, but we use these expressions to talk about constructions. To appreciate this point, consider the construction (x ranging over τ)

$$\lambda x\ [{}^0\!> x\ {}^0 0].$$

As a *lambda term* it contains brackets, the sign 'λ', two occurrences of the letter *'x'*, . The respective *construction* does not contain brackets, nor does it contain any letters or signs, but it does contain one occurrence of the variable x. The seeming occurrence of *'x'* in 'λx' is our way of saying that the construction abstracts over x.

It is very important to realize that constructions are not reducible to set-theoretic objects[10]. A simple example: The type of – ('minus'), 3, 5 is $(\tau\tau\tau)$, τ, τ, respectively. Consider the construction

$$[{}^0\!-\ {}^0 5\ {}^0 3]$$

and the set

$$\{{}^0\!-\ ,\ {}^0 3,\ {}^0 5\}.$$

The latter is a set containing the simple parts of the former: it evidently differs from the construction. As Zalta in his [32] says:

> [A]lthough sets may be useful for describing certain structural relationships, they are not the kind of thing that would help us to understand the nature of presentation. There is nothing about a set in virtue of which it may be said to present something to us.

[9] A mode of presentation (construction) of an object could not be distinguished from the object itself, thus underpinning the very point of distinguishing between a mode of presentation and objects so presented.

[10] Notice that all objects whose type is of order 1 are set-theoretic objects.

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Notice that the same set can be connected with the construction

$$[^0\!- \ ^0 3 \ ^0 5]$$

which constructs another object than the first one.[11]

Having defined constructions, we can now define *higher-order types.* The inductive definition that captures the whole *ramified hierarchy of types* proceeds in three steps: first, types of order 1 are defined (see above), then *constructions of order n*, and finally the types of order $n + 1$.

Definition *Higher-order types (ramified hierarchy)*

$\mathbf{T_1}$ (see above)

$\mathbf{C}_n$ Let α be a type of order n.

i) Let a variable ξ range over α. Then ξ is *a construction of order n.*
ii) Let X be an object of type α. Then ^{0}X is *a construction of order n.*
iii) Let X, X_1,...,X_m be *constructions of order n.* Then [X X_1...X_m] is *a construction of order n.*
iv) Let x_1,...,x_m, X be *constructions of order n.* Then [λx_1...x_m X] is *a construction of order n.*
v) Nohing other...

$\mathbf{T}_{n+1}$ Let $*_n$ be the collection of all constructions of order n.

i) $*_n$ and all types of order n are *types of order* $n + 1$.
ii) Let α, β_1,...,β_m be *types of order* $n + 1$. Then the set of partial functions ($\alpha \ \beta_1 \ldots \beta_m$) is a *type of order* $n + 1$.
iii) Nothing other...

Comments:

i) The stipulation in T_{n+1} "and all types of order *n...*" is necessary to handle the cases where not all constructions in a composition or closure are of the same order. Then the point i) in T_{n+1} says that what counts is the highest order ('type raising').

ii) Now we can distinguish the type of an object (including constructions) and the type of the constructed object. Let A be an object that belongs to the type α. We will write

$$\mathrm{A} \ / \ \alpha.$$

Let A be a construction that (v-)constructs an object of the type α. We write then

[11] Bolzano seems to have sensed that there is a fundamental distinction here. See [3, §244], mentioned above.

$$A \rightarrow \alpha.$$

To illustrate this distinction, let us consider a numerical variable, say, x_1. We could write

$$x_1 \rightarrow \tau$$

but since x_1 is a construction of order 1 (see C_n i)) it belongs to the type $*_1$, so we write

$$x_1 \ / \ *_1.$$

Now, when we want to (or in a deductive process must) *mention* this variable, we write '${}^0x_1$'. This trivialization constructs x_1, hence an object of the type $*_1$, hence an object of the type of order 2 (see T_{n+1} i)) and we write

$${}^0x_1 \rightarrow *_1$$

and

$${}^0x_1 / \ *_2$$

etc. Trivialization increases the order of a type.

iii) If an expression E denotes an object of type α, we will say that E is also of type α, denoted E:α.

Empirical expressions denote intensions, $\alpha_{\tau\omega}$-objects. We will use variables *w, t* as ranging over possible worlds ω and times τ. A composition of a construction C of an α-intension A with variables w, t, i.e., $[[C\ w\ ]\ t\ ]$ (the intensional descent of A), will be often abbreviated as C_{wt}.

Now we will define two ways of binding variables. We do it *via* defining free and bound *occurrences* of variables (the rest is definable as usual):

Definition *Free, λ-bound, ^{0}-bound variables:*
Let C be a construction containing at least one occurrence of the variable ξ.

i) If C is ξ then the occurrence of ξ in C is *free in C.*

ii) If C is ^{0}D (for some construction D) then every occurrence of ξ in D (= in C) is *^{0}-bound* ('trivialization-bound') *in C.*

iii) If C is $[\lambda x_1 \ldots x_m\ X]$ then every occurrence of ξ identical with one of $x_1,\ldots,x_m$ is λ*-bound in C* unless it is *^{0}-bound* in some component of X.[12]

iv) If C is $[X\ X_1 \ldots X_m]$ then every occurrence of ξ in C which is not λ- or ^{0}-bound is *free in C.*

[12]The notion of *component* or *subconstruction* is explained, e.g., in [19]. The intuitive sense should be obvious.

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The two ways of binding share one intuitive property because of which the respective variables are called *bound*: A bound variable is inaccessible in the sense that it is immune w.r.t. a valuation. A variable which is ^{0}-bound is not used, it is *mentioned*. Two constructions that differ only in containing distinct λ-bound variables (due to a correct renaming) obey the α-rule of λ-calculi, i.e., they are equivalent. No analogy holds for the ^{0}-bound variables. For an example, compare the following two cases:

I. $$\lambda x_1[{}^0\!+\ x_1\ {}^0 1],\ \lambda x_2[{}^0\!+\ x_2\ {}^0 1]$$

II. $${}^0[\lambda x_1[{}^0\!+\ x_1 {}^0 1]],\ {}^0[\lambda x_2[{}^0\!+\ \mathrm{x}_2\ {}^0 1]]$$

The equivalence of the I. pair is obvious. The second pair represents *non-equivalent* constructions: they construct two distinct, albeit equivalent constructions. (Yet they do construct one and same *concept*, see later.)

Clearly, particular constructions can be created due to two simple rules so that the trees generated in this way are analogous to the trees characteristic of categorial grammars. The rules are:

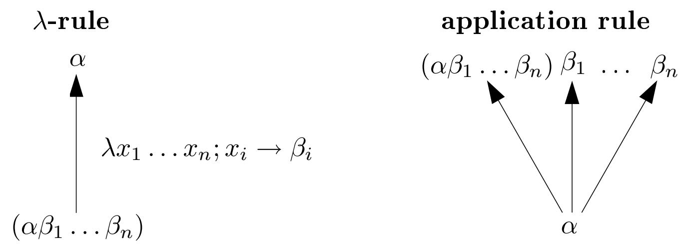

Figure 1: Rules of creating constructions

Any such tree leads to a construction after all its terminal nodes are occupied by constructions that (v-)construct an object of the respective type.

In Figure 2 we adduce an example of a 'type-theoretical tree' (TTT) considering the sentence *Charles calculates 56 - 18.*

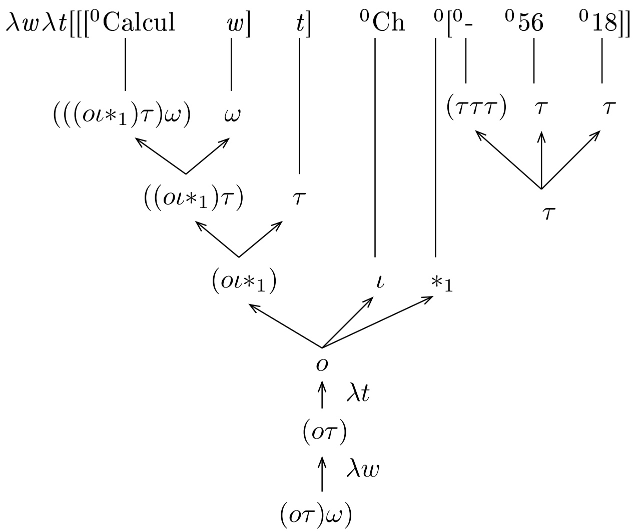

Figure 2: Type-theoretical tree

First, we have to determine types of the objects denoted by meaningful subexpressions of the sentence, and we will adduce some arguments for accepting the proposed types. 'Charles' is a name of some particular individual. Suppose that this individual is determined by some context. So we have[13] Ch(arles)/ι. 'Calculate', being an empirical expression, denotes the following function (intension): in a given world W it returns a chronology (i.e., a function from time moments, T) of the relation between an individual and a numerical construction. So we have Calcul(ate)/ $(((o\iota{*_1})\tau)\omega)$, (abbrev. $(o\iota{*_1})_{\tau\omega}$). *56 – 19* is the name of the construction $[^0 - {}^0 56\ {}^0 18]$ of the type $*_1$. '–' is a name of the operation *substraction*, type $(\tau\tau\tau)$. '56' as well as '18' are names of numbers, so the type is τ. The whole construction constructs a number but the construction is *mentioned* in the expression so that we have

$$\lambda w \lambda t\ [[[{}^0\text{Calcul}\ w]\ t]\ {}^0\text{Charles}\ {}^0[{}^0- {}^0 56\ {}^0 18]],$$

which can be abbreviated

$$\lambda w \lambda t\ [{}^0\text{Calcul}_{wt}\ {}^0\text{Charles}\ {}^0[{}^0- {}^0 56\ {}^0 18]].$$

The correspondence between the types, terminal nodes of the TTT, and brackets is obvious.

The example shows, however, that the way the objects have been associated with types is not, in general, satisfactory. Here the tree associated with the construction $[^0 - {}^0 56\ {}^0 18]$ has been ignored, and rightly so, since the construction

[13]For the sake of simplicity, we will consider proper names as 'labels' of individuals.

has been *mentioned* so that the only type that counts is $*_1$. All the same, full ignorance would lead to non-reasonable constructions in some cases. To see this, let us consider the intension *calculate*. Its type is $(o\iota *_1)_{\tau\omega}$, which means that it is a relation-in-intension between individuals and constructions. We call this intension *calculate* to suggest that the construction to which the respective individual is linked is a construction of some (arithmetical or so) calculation. If we totally ignored this character of the mentioned construction we could associate the terminal node with mentioning any construction of order 1, for example the construction [^{0}Color ^{0}Blue], which constructs **T**, since the property (*being*) *blue* is a member of the class of colors. Clearly, if the type of *calculate* is $(o\iota *_1)_{\tau\omega}$ and no other support of our intuition as regards the character of this relation is given, then we would have to admit that one can calculate (in the arithmetical sense) the construction [^{0}Color ^{0}Blue], which is surely absurd. Thus, when a type contains a higher-order subtype, additional type-theoretical information has to be filled in to determine the type of the object constructed by the respective higher-order type. In our case we can write

$$\text{Calculate} \ / \ (o\iota *_1)_{\tau\omega},\ *_1 \to \tau$$

This move is not sufficient either. If the respective construction were, e.g., ^{0}3, the result would be nonsensical: we cannot calculate a particular number. So a general schema of arithmetical operations of any complexity should be adduced. A preliminary finding is that

> *any well-defined construction leads unambiguously to a TTT but not every TTT leads to a well-defined construction.*

4 Analysis of expressions, logical form of expressions.

As regards our problem of *logical form,* all this means that a logical form of a meaningful expression is always attainable. It can happen, of course, that even a quasi-expression lacking any meaning can have a logical form assigned to it, viz. if its logical form is conceived of as being determined by a TTT.

Let us reinterpret the preceding example. We have claimed that the terminal nodes of the TTT were some extra-linguistic objects, viz. constructions. We have named those objects to make their intended character obvious. Now we will consider an expression and try to connect it with a construction that could be construed as being its analysis. Intuitively, we can imagine that somewhere in the background we have some Montague-like rules at our disposal. To make our task easier we recycle the previous example.

So let our expression be

(E) *Charles calculates 56 – 18.*

The first step to be done is the *type-theoretical analysis.* It consists in associating every simple meaningful subexpression of E with the type of its denoted object. This part of our work can in principle be covered by rules similar to Montague's that generate types dependently on the grammatical category of the expression. We follow an underlying principle that empirical expressions denote non-trivial intensions. In the preceding chapter we assigned types to the objects talked about by (E) as follows:

$$\text{Charles} \mathbin{/} \iota,\ \text{Calcul(ate)} \mathbin{/} (o\iota *_1)_{\tau\omega},\ [^0 - {}^056\ {}^018] \mathbin{/} *_1.$$

The second step could be called *synthesis.* Having at our disposal objects of the types ι, $(o\iota*_1)_{\tau\omega}$, $*_1$, composing their constructions (and using variables) we have to find a construction which constructs a *proposition*, i.e., an $o_{\tau\omega}$-object.

Our example is not difficult to solve. We have to apply the intension *calculate* first to a possible world and then the result of this application to an instant of time. The resulting construction v-constructs a relation between individuals and constructions of order 1. The individual is constructed by the trivialization of Charles, and the construction is constructed by the trivialization of $[^0 - {}^056\ {}^018]$. The problem is: where should possible worlds and times be taken from? Fortunately, we have variables at our disposal. Let w be a variable ranging over ω, and t be a variable ranging over τ. Then we have

$$\underbrace{[[^0\text{Calcul } w]\ t]}_{(o\iota*_1)}$$

and we need a pair $\langle$individual, construction of order 1$\rangle$, which is constructed, respectively, by ^{0}Charles and $^0[^0 - {}^056\ {}^018]$. Thus

$$[\underbrace{[[^0\text{Calcul } w]\ t]}_{(o\iota*_1)}\ \underbrace{^0\text{Charles}}_{\iota}\ \underbrace{^0[^0\text{-}\ {}^056\ {}^018]}_{*_1}]$$

$$\begin{tikzcd} (o\iota*_1) & \iota & *_1 \\ & o \arrow[ul] \arrow[u] \arrow[ur] & \end{tikzcd}$$

The resulting construction v-constructs a truth-value; whether it is **T**, or **F** is determined by the valuation for w, t. We know, however, that our sentence — being empirical — cannot denote a truth-value. It denotes a proposition, and the respective proposition is constructed *via* an abstraction over τ and ω. Two applications of the λ-rule yield

(C) $\lambda w \lambda t\ [[[^0\text{Calcul } w]\ t]\ {}^0\text{Charles}\ {}^0[^0 - {}^056\ {}^018]]$, or, abbreviated as $\lambda w \lambda t\ [{}^0\text{Calcul}_{\text{wt}}\ {}^0\text{Charles}\ {}^0[{}^0 - {}^056\ {}^018]]$.

$$\begin{array}{c} \vdots \\ o \\ \uparrow\ \ \lambda t \\ (o\tau) \\ \uparrow\ \ \lambda w \\ ((o\tau)\omega) \end{array}$$

From the viewpoint of TIL, (C) is a procedure that can be conceived of as the *meaning* of (E).

Now we are going to show that any meaningful expressions E has a *logical form* that can be derived from its meaning, i.e., from the construction expressed by E. Consider the meaning of the sentence (E). The construction C can be modified as follows: First, trivialisations of the objects (E) talks about, i.e., ^{0}Calcul, ^{0}Charles and ${}^0[{}^0 - {}^056\ {}^018]$ are substituted for by *linguistic variables* (not in the sense of Zadeh's use of this term), i.e., variables for which those expressions can be substituted which denote objects of the same type as the type of the object constructed by the respective trivialisation. Second, the variables (*w, t* in our example) remain as they are. Third, for the linguistic variable corresponding to $[{}^0 - {}^056\ {}^018]$ (the type $*_1$) only a name of arithmetical operation can be substituted. The resulting concatenation of the linguistic symbols is

(F) $$\lambda w \lambda t\ [[[\text{X}\ w]\ t]\ \text{Y Z}],$$

or the abbreviated

$$\lambda w \lambda t\ [\text{X}_{\text{wt}}\ \text{Y Z}],$$

where the types of substitutable expressions are X: $(o\iota *_1)_{\tau\omega})$, Y: ι, Z: $*_1$.

Let us return to the expression (E):

Charles calculates 56 – 18.

Intuitively, the logical form of (E) from the viewpoint of truth-functional logic would be, say,

$$p.$$

1$^{\text{st}}$-order predicate logic would propose

$$\text{P(c)},$$

where 'P' is a monadic predicate symbol and 'c' an individual constant. Perhaps a more fine-grained option would be

(*) $$\text{R(c, f(a, b))}$$

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with 'R' as binary predicate symbol, 'f' a binary functional symbol and 'a', 'b' individual constants. But the fact that calculating concerns a procedure rather than its outcome cannot be captured by this proposal. The elementary consequence of (*) has an unacceptable result:

$$\exists x\ (\mathrm{R}(\mathrm{c}, x)\ \&\ (x = \mathrm{f}(\mathrm{a}, \mathrm{b}))).$$

By substituting the respective constants we can obtain a meaningless sentence namely that "Charles calculates 38": in our case, Charles does not calculate the x which is the result of 56 – 18.

Besides, the presence of the empirical factor (intensionality) is completely ignored. Our analyses show, however, that intensionality should be visible in the logical form. So let us consider (F):

$$\lambda w \lambda t\ [\mathrm{X}_{\mathrm{wt}}\ \mathrm{Y}\ \mathrm{Z}].$$

This is a quasi-expression and any substitution leads again to a quasi-expression: it is always a hybrid that contains normal expressions of the given language and signals such as brackets or 'λ', not to mention the strange bound variables w, t. Indeed, consider one result of a correct substitution

$$\lambda w \lambda t\ [\mathit{Calculates}_{\mathrm{wt}}\ \mathit{Charles}\ \mathit{56} - \mathit{18}]$$

and compare it with (E):

Charles calculates 56 – 18.

Well, word-order and suchlike can be dealt with by means of some transformational procedures. In any case, the schema (F) resembles the syntactic form of some artificial language (so it could be admitted as a candidate for *logical form*), but actually it is a schema of the fixation of a procedure determined by the corresponding TTT. Our question is: *Why should such schemas* (*quasi-expressions*) *not be allowed to represent logical forms*? They obviously satisfy the expectations connected with the notion of logical form. Moreover, they are much more informative than other candidates obtained by using less-expressive formal logics. For instance, from (E) we can infer that Charles calculates something, not however the number 38, but the operation 56–18. The form (F) renders this fact in the correct way: variable Z is of type $*_1$. Schemas of the type (F) are, of course, not schemas that would capture *grammatical forms*: they have to capture *logical forms*, i.e., to decipher the code underlying the grammatical form.

Besides, schemas of the type (F) are sensitive to the degree of grammatical granularity. As an example, let us consider the following modification of our sentence (E):

(E') *The richest man calculates 56 – 18.*

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Analyzing (E) we have presupposed that *'Charles'* is a rigid designator in the sense that the individual named *Charles* here is the same in all possible worlds.[14] This assumption does not hold in the case of *'the richest man'*. Our way of creating schemas like (F) is connected with the requirement that the variables (in the terminal nodes) are substitutable only by expressions whose type is the type of the terminal node. Now associating (E') with (F) would not satisfy this requirement: the type of *the richest man* is no more ι. Moreover, this expression consists of two meaningful subexpressions, *'the richest'* and *'man'*[15]. Our search for the relevant logical form must take this fact into account. So our type-theoretical analysis will replace *Charles* / ι with *the richest* / $(\iota(o\iota))_{\tau\omega}$ (the function that chooses, in the given world-time, the individual — if any — who is the richest among the members of the given class) and *man* / $(o\iota)_{\tau\omega}$. We get the type-theoretical tree and the respective construction illustrated by Figure 3.

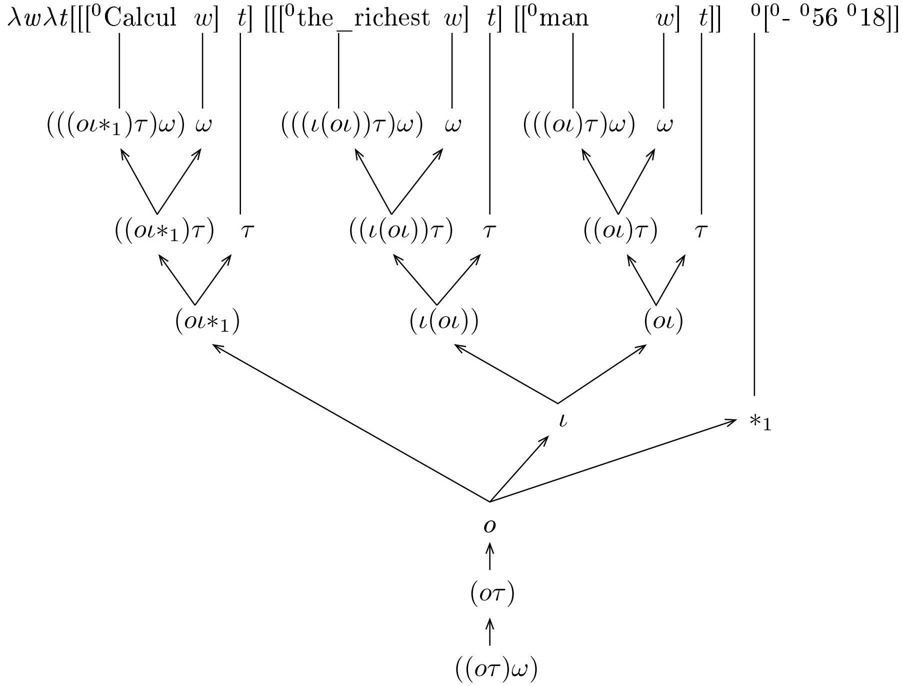

Figure 3: Type-theoretical tree of (E')

Now the logical form will be

[14] The problem of proper names is much more complicated, of course, but for our present purpose our simplification is innocuous.

[15] The list of subexpressions with a self-contained meaning has to be provided by a linguist, of course. In this case, the subexpression 'the richest man' could be decomposed into: 'the most', 'rich', 'man'.

(F') $\lambda w \lambda t\ [X_{wt}\ [Y_{wt}\ Z_{wt}]\ U]$

with X: $(o\iota *_1)_{\tau\omega}$, Y: $(\iota(o\iota))_{\tau\omega}$, Z: $(o\iota)_{\tau\omega}$, U: $*_1$.

Remark 1: The type $(\iota(o\iota))_{\tau\omega}$ is the type of empirical individual singularisers. They are frequently named by superlatives such as 'the tallest', 'the richest', and they are functions which in a given possible world W and time T select one individual from a set of individuals, namely the tallest, or the richest one, if there is such an individual, otherwise they do not have any value in W at T. Sentences containing a definite description such as 'the tallest man', 'the richest man' in the *de re* supposition[16] have an existential presupposition. In our case, if there is no richest man, the sentence (E') does not have *any* truth value. Hence the sentence (E'), unlike the sentence (E), has an existential presupposition, which is expressed by the respective logical form: the composition corresponding to $[Y_{wt}Z_{wt}]$ can fail to return any value.

Remark 2: The notion of logical form, as we introduced it above, is a fine-grained notion: all the arguments of the same form as a valid argument are valid. The question is whether this notion is not too fine-grained. For instance, in our case we could suppose that logical forms differing only in containing constructions of a type $*_n$ of any order n instead of $*_1$ were conceived as one and the same form. Thus, the sentence "Charles is proving that dividing 5 by 0 is improper" would receive the form (F), though *proving* is here of type $(o\ \iota\ *_2)_{\tau\omega}$ and the meaning of "dividing 5 by 0 is improper" is a construction of order 2. Where *Improper* is a set of improper constructions of order 1 and *Divide* is a function of type $(\tau\tau\tau)$, the analysis becomes: $[{}^0\text{Improper}\ {}^0[{}^0\text{Divide}\ {}^0 5\ {}^0 0]]$, which is of the type $*_2$.

5 The Parmenides Principle. Conceptual systems

Consider the sentence

(G) *The biggest planet*[17] *is smaller than the sun.*

and compare it with the sentence

(G') *Jupiter is smaller than the sun.*

This gives rise to two questions:

1) Is (G) equivalent to (G')?

2) Which objects do the sentences (G) and (G') talk about?

[16] For details on *de re* vs. *de dicto* supposition see, e.g., [10], [28].
[17] By 'planet' we mean here the major planets of our solar system

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The answer to 1) is *No* and is justified as follows. Since it is contingent that Jupiter plays the role of the biggest planet, there are possible worlds and times where Jupiter is not the biggest planet. In some of these worlds the biggest planet is not smaller than the sun, but Jupiter — the same individual in all worlds and times — is still smaller than the sun. Thus (G) is, and (G') is not, false in such worlds.

As for 2), (G') does, and (G) *does not,* talk about Jupiter. To this point we will return immediately. Second, (G) talks about the following (abstract!) objects: (a) *the biggest,* (b) *planet,* (c) *smaller than,* (d) *the sun,* (e) *the biggest planet,* (f) *smaller than the sun,* and, finally, about the proposition (g) *that the biggest planet is smaller than the sun.* The respective types are:

(a) B(iggest)/$(\iota\ (o\iota))_{\tau\omega}$ (superlative, see the preceding example)

(b) P(lanet)/$(o\iota)_{\tau\omega}$

(c) Sm(aller than)/$(o\iota\iota)_{\tau\omega}$

(d) S(un)/ι (taking "Sun" as a proper name)

(e) BP/$\iota_{\tau\omega}$ (the biggest planet)

(f) SmS/$(o\iota)_{\tau\omega}$ (smaller than the sun)

(g) BPSS/$o_{\tau\omega}$ (the proposition that the biggest planet ...)

(G) does not talk about Jupiter, (G') ignores a), b), e).

We do not share the widespread conviction that (G), talking about the biggest planet *eo ipso* talks about Jupiter.[18] Simple counterexamples can show that this identification is not reasonable. For example: the expression *the biggest planet* has been introduced — as a well formed expression — into English independently of the contingent fact that Jupiter satisfies this role. To know which planet is the biggest one we have to know the meaning of *the biggest*, of *planet* and the meaning of their concatenation. To know what Jupiter is we need not know these meanings. The sentence *Jupiter is the biggest planet* is not an analytic statement, it claims an empirical truth holding only in some worlds-times. He who says "Jupiter is not the biggest planet" shows that (s)he does not know some facts about our real world. The same speaker would probably never say "Jupiter is not Jupiter", since this would mean that his/her *reasoning* is not correct.

It was Frege in his [12, p. 60] who formulated a warning concerning this prejudice:

> *Ueberhaupt ist es unmöglich, von einem Gegenstande zu sprechen, ohne ihn irgendwie zu bezeichnen oder zu benennen.*[19].

[18] Many places where Tichý criticises this prejudice can be found in his [31].

[19] "It is not at all possible to talk about an object unless it is in some way denoted or named." Unfortunately Frege himself did not obey this principle when he thought that speaking about the morning star we speak about Venus

This principle (called by Tichý the *Parmenides Principle* in an unpublished study) says that *what is not denoted by an expression E cannot be talked about.* So when we say "(All) whales are mammals" we do not say anything about *particular* whales, since they are not denoted in the sentence. When saying "The highest mountain is in Asia" we are not talking about Mount Everest. If another mountain were (or became) the highest one and were in Africa our sentence would be false, whereas the sentence "Mount Everest is in Asia" would remain true. When speaking about the current president of the U.S.A., we are not speaking about George W. Bush (G.W. Bush surely wanted to become the current president of the U.S.A., but obviously this does not mean that he wanted to become G.W. Bush).

This principle can be reformulated so that it concerns our constructions. Let C be a construction that is associated with an expression E as its meaning. Our principle says then that *every component of C, with the exception of bound variables, must be associated with some subexpression of E as its meaning.* Notice that both our attempts at analyzing expressions *via* constructions obeyed this principle.

Now we analyze the sentence (G) using the trivializations of B, P, Sm, S above. When drawing the type-theoretical tree, we will use notational abbreviations. Thus, where A is a construction of an α-intension of type $\alpha_{\tau\omega}$, we will write simply

$$\begin{array}{c} \mathrm{A}_{\mathrm{wt}} \\ | \\ \alpha \end{array}$$

(G) predicates the relation 'smaller than' (Sm) of the individual playing the role of the biggest planet and of the sun, in that order. Thus, an admissible analysis can be for instance:

C $\lambda w \lambda t\ [{}^0\mathrm{Sm}_{wt}\ [\lambda w \lambda t\ [{}^0\mathrm{B}_{wt}\ {}^0\mathrm{P}_{wt}]]_{\mathrm{wt}}\ {}^0\mathrm{S}]$, or β-reduced

C1 $\lambda w \lambda t[{}^0\mathrm{Sm}_{\mathrm{wt}}\ [{}^0\mathrm{B}_{\mathrm{wt}}\ \ {}^0\mathrm{P}_{\mathrm{wt}}]\ {}^0\mathrm{S}]$

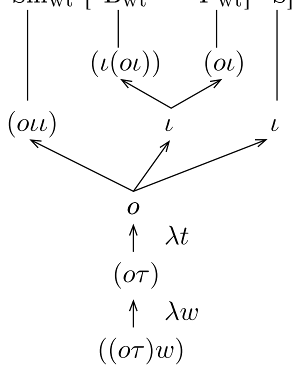

Figure 4: Type-theoretical tree of the sentence (G)

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In the following analyses we will omit the TTTs; their derivation is easy — it can be recommended as a type-checking method. Now we first show that substituting ^{0}J(upiter) for $[\lambda w \lambda t\ [^0\mathrm{B}_{wt}\ ^0\mathrm{P}_{wt}]]_{\mathrm{wt}}$ in **C** (or for $[^0\mathrm{B}_{wt}\ ^0\mathrm{P}_{wt}]$ in **C1**) does not satisfy our principle.

$$\lambda w \lambda t[^0\mathrm{Sm}_{\mathrm{wt}}\ \underset{\iota}{\underset{|}{^0\mathrm{J}}}\ ^0\mathrm{S}]$$

^{0}J cannot be associated with *'the biggest planet'* as its meaning, since *'the biggest planet'* denotes an individual role of type $\iota_{\tau\omega}$, whereas *'Jupiter'* denotes an individual. ^{0}J, $\lambda w \lambda t\ [^0\mathrm{B}_{wt}\ ^0\mathrm{P}_{wt}]$ construct two different objects, even of different types, that coincide at some W, T, but not all.

Let us say that *an expression E talks about an object O according to the construction C if C, which is associated with E as its meaning, or any subconstruction of C, constructs O.* Which objects does the sentence

The biggest planet is smaller than the sun

talk about according to **C1**? Obviously, it talks about the objects B, P, Sm, S. Further, of course, it talks about BPSmS (the proposition). There are, however, two other objects that we suggest the sentence talks about: BP and SmS. The object BP can be constructed either by the trivialization or by the composition of ^{0}B and ^{0}P, abstracting over w, t:

$$^0\mathrm{BP},\ \lambda w \lambda t\ [^0\mathrm{B}_{wt}{}^0\mathrm{P}_{wt}].$$

So we can offer two other constructions as the meaning of our sentence:

C2 $$\lambda w \lambda t\ [^0\mathrm{Sm}_{wt}\ ^0\mathrm{BP}_{wt}\ ^0\mathrm{S}]$$

or

C3 $$\lambda w \lambda t\ [^0\mathrm{Sm}_{wt}\ \lambda w \lambda t[^0\mathrm{B}_{wt}\ ^0\mathrm{P}_{wt}]_{wt}\ ^0\mathrm{S}].$$

A comparison with **C1**: According to **C2** the sentence talks about Sm, S, BP, and the proposition, **C3** talks about Sm, S, B, P, BP, and the proposition. What about the object SmS? This is a property of individuals. It can be constructed as follows, $x \rightarrow \iota$:

$$\lambda w \lambda t\ [\lambda x\ [^0\mathrm{Sm}_{wt}\ x\ ^0\mathrm{S}]].$$

Thus, the sentence can be also seen as predicating this property of the biggest planet. This way we have

C4 $$\lambda w \lambda t\ [[\lambda w \lambda t\ \lambda x\ [^0\mathrm{Sm}_{wt}\ x\ ^0\mathrm{S}]]_{wt}\ [^0\mathrm{B}_{wt}{}^0\mathrm{P}_{wt}]]$$

C5 $$\lambda w \lambda t\ [[\lambda w \lambda t\ \lambda x\ [^0\mathrm{Sm}_{wt}\ x\ ^0\mathrm{S}]]_{wt}\ ^0\mathrm{BP}_{wt}]$$

C6 $$\lambda w \lambda t\ [[\lambda w \lambda t\ \lambda x\ [^0\mathrm{Sm}_{wt}\ x\ ^0\mathrm{S}]]_{wt}\ \lambda w \lambda t\ [^0\mathrm{B}_{wt}{}^0\mathrm{P}_{wt}]_{wt}].$$

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Finally, in theory we can construct the whole proposition as follows:

C7 $\quad {}^0$BPSmS.

Now we can compare particular constructions according to the objects constructed by their subconstructions. (We do not need to consider **C7**, which is obviously the worst analysis.)

Analyses	*Objects talked about by the sentence according to the construction*
C1	BPSmS, Sm, B, P, S
C2	BPSmS, Sm, BP, S
C3	BPSmS, Sm, B, P, BP, S
C4	BPSmS, Sm, B, P, SmS, S
C5	BPSmS, Sm, BP, SmS, S
C6	BPSmS, Sm, B, P, BP, SmS, S

We can see that the proposed analyses differ in their ability to capture the objects the sentence talks about. Our problem consists in ordering these analyses according to some general criterion, to make it possible to single out, in general, an analysis (if any) that would be so fine-grained as to make it possible to derive all the consequences of the sentence. If such an analysis exists then this analysis (with respective variables substituted for trivializations) would be the best candidate for being the optimal logical form. Two such criteria — not independent — can be formulated (see [20]).

Criterion 1. A construction C is a *worse analysis of an expression E than a construction* C' iff the set of correct inferences based on C is a proper subset of the set of correct inferences based on C'.

Criterion 2. A construction C is a *worse analysis of an expression E than a construction* C' iff the set of those subexpressions of E that are associated (in harmony with the Parmenides Principle) with a component of C is a proper subset of the set of subexpressions of E that are associated with C'.

Criterion 1 is obviously prior to criterion 2. However, it cannot be checked effectively, unlike criterion 2. So when searching for the best analysis, we use criterion 2 and try to compose constructions of all the objects the expression talks about.

We could say that **C2** is worse than **C3**, because from **C3**, unlike **C2**, the consequence that *some planet is smaller than the sun* is derivable (*ceteris paribus*). **C1** is also worse than **C3**, because the fact that the *concept* (see below) of the biggest planet is used *de re* in our sentence is derivable from **C3** but not from **C1.** However, **C2** and **C1** are not comparable, because **C1** neglects the concept of the biggest planet BP and **C2** neglects the concepts of B and P.

According to criterion 2 it might seem that **C4** is better than **C1, C5** is better than **C2,** and finally, **C6** better than **C3,** because they talk about the property SmS that is ascribed to the biggest planet. However, it is not the case. To see why, compare **C6** and **C3** according to criterion 1: we can easily see that from **C3** the fact that the property SmS is ascribed to the biggest planet is derivable by η-expansion. Similarly for **C4**, **C5**, these analyses are redundant according to criterion 1. Thus **C4**, **C5**, **C6**, though being more fine-grained than **C1**, **C2**, **C3**, are not optimal; as the optimal analysis we choose the one that does not use η-expanded constructions (which are not concepts, see [15]) of the objects talked about.

Moreover, we are looking for *the* best analysis. According to criterion 2, we would find *two* equally good analyses which are not comparable. The sentence (G) talks also about another property BPSm, namely 'being an x that the biggest planet is smaller than the x'. Taking BPSm into account, we would obtain three other constructions, **C8**, **C9**, **C10**, the best of which is:

C10 $\qquad \lambda w\lambda t\ [[\lambda w\lambda t\ \lambda x\ [{}^0\mathrm{Sm}_{wt}\ \lambda w\lambda t[{}^0\mathrm{B}_{wt}\ {}^0\mathrm{P}_{wt}]_{wt}\ x]]_{wt}\ {}^0\mathrm{S}]$.

However, **C10** is not comparable to **C6** according to criterion 2, and we would obtain two maximally good constructions. This analysis is again not the optimal one: the property BPSm can be easily derived from **C3** by η-expansion.

The moral of this story is: when looking for the best analysis, criterion 1 is fundamental, though not effective. Therefore, we use the auxiliary criterion 2 to assign to *each* meaningful subexpression a construction called *concept* (see below), i.e. the minimal *not* η-expanded construction of the denoted object, so that constructions redundant according to criterion 1 are avoided.

Let the reflexive closure of the relation defined in criterion 1 be denoted '$\leq$'. Obviously, $\leq$ is a partial ordering, and it can be proved that the poset $\{\mathbf{C}, \leq\}$, where **C** is the set of possible analyses of the given expression, can be ordered in such a way that a complete lattice arises whose greatest element would be the best analysis.

The resulting lattice given by **C1**, **C2**, **C3**, **C7** according to this ordering is illustrated by Figure 5.

In this Hasse diagram we have also marked the redundant constructions **C4, C8, C5, C9, C6, C10** that are better than the respective **C1, C2, C3** according to criterion 2 though not according to criterion 1.

The Parmenides Principle is frequently in harmony with our linguistic intuitions, which have been a little bit spoiled under the influence of formalizations. One example:

Some dogs are dangerous.

Our 'logical training' makes us believe that the following analysis should be O.K.:

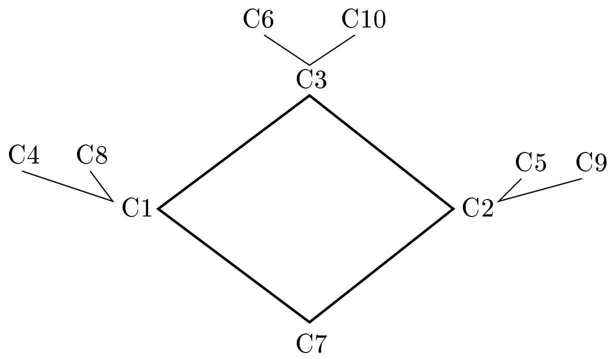

Figure 5: Hasse diagram of ordering the analyses of the sentence (G)

Types: Some ($= \exists$) / $(o(o\iota))$
dog (D)/ $(o\iota)_{\tau\omega}$
dangerous (Dn) / $(o\iota)_{\tau\omega}$
$\wedge$ / (ooo) (!)

Synthesis: $\lambda w \lambda t\ [^0\exists\ \lambda x\ [^0\wedge\ [^0\mathrm{D}_{wt}\ x]\ [^0\mathrm{Dn}_{wt}\ x]]]$

The Parmenides Principle has been violated, since our sentence contains no reference to conjunction. The construction could be an analysis of the sentence

Some individuals are dogs and dangerous (objects).

Sainsbury in his [26, p. 328], speaks about "the absurd claim that an English quantification contains an invisible occurrence of 'if' or '→'", as in "All dogs are predators". Or (p.333) he derides "exotic logical form proposals, for example the view that English universal quantifications are quantifications of conditionals". Yet later we will see that in some cases 'inserting' conjunctions, etc., cannot be classified as a violation of the Parmenides Principle.)

The remedy to the violation is simple. Quantifiers can be retyped as $((o(o\iota))(o\iota))$. Thus the existential quantifier, when applied to a class A, returns the class of those classes which share at least one member with A. In our case, the analysis of the sentence

Some dogs are dangerous.

will be

$$\lambda w \lambda t\ [[^0\exists\ ^0\mathrm{D}_{wt}]\ ^0\mathrm{Dn}_{wt}].$$

Similarly, the general quantifier, when applied to a class A, returns the class of all classes which contain A as their subset. The analysis of the sentence *"All dogs are predators"* will be:

$$\lambda w \lambda t\ [[{}^0\forall\ {}^0\mathrm{D}_{wt}]\ {}^0\mathrm{Pr}_{wt}].$$

Now it seems as if the Parmenides Principle would enable us to easily solve the old problem of the 'ideal semantic analysis of expressions of natural language'. It must be clear, however, that such an easy solution is at least highly suspect. For instance, we neglected the problem of building up a system of rules (that would replace our intuitive steps), the problem of handling irregularities of any natural language, and of taking into account special scientific languages that use such terms whose analysis would require other atomic types. We mention here a special problem defined as follows:

Our method consisted in isolating particular meaningful subexpressions of the given expression and associating them with adequate types. The atomic building stones of this activity were simple expressions, mostly single words. Any simple expression A was construed as denoting an object, say, A. Thus the construction that should represent the meaning of A was simply the trivialization ^{0}A. The assumption that was presupposed when this method was used is, however, false. To show this we need some auxiliary definitions. (See, e.g.,[9], [17, §5], [19, p. 42].)

Closed constructions:

A construction is *closed* iff it does not contain any free variable.

Concepts:

A *concept of order n* is a closed construction of order *n*.

Examples:

a) ^{0}Prime

b) $\lambda x\ [{}^0\geq\ x\ {}^00]\ (x \to \tau)$

c) $\lambda w \lambda t\ [{}^0\exists\ \lambda x\ [{}^0\wedge\ [{}^0\mathrm{Mammal}_{wt}\ x][{}^0\mathrm{Live\_in\_water}_{wt}\ x]]]\ (x \to \iota)$

Comments:

1. Questions like the following ones can easily arise:

Is b) another concept than b') $\lambda y\ [{}^0\geq\ y\ {}^00]\ (y \to \tau)$?

Is a) another concept than a') $\lambda x\ [{}^0\mathrm{Prime}\ x]$?

The procedural theory of concepts based on TIL construes concepts as abstract procedures, concepts are not reducible to set-theoretical objects[20] (which follows from the fact that they are a kind of constructions). The abstract procedure behind a) does not essentially differ from that one behind a'); the same holds for b) and b'). One attempt to take this fact

[20] We accept Church's conception according to which the meaning of an expression is a *concept* of the denotation (see [6, §01]).

into account has been made in [17], where the author defined concepts as equivalence classes induced by a similarity relation Quid ('quasi-identity'). A fundamental objection to this solution can be (and has been) raised: if concepts are procedures (unlike sets) then they cannot be defined as equivalence *classes*. In [15] another definition is proposed, according to which concepts are those members of the equivalence class which are results of a normalization procedure applied to any member of that class. (The other members of it are said to 'point to' the concept.) This second solution makes it possible to retain the definition of concepts as closed constructions.

2. Consider now a). The procedure ^{0}Prime consists in constructing the class of prime numbers 'directly', without using other concepts. This particular example invites suspicion. Does the meaning of the word *'prime'* consist in a simple procedure that takes the infinite class of prime numbers and enables us to know every member of it (NB without using such concepts as to be divisible, etc.)? In other words, do we understand the word *'prime'* because we are able to grasp the infinite class of primes without mediation of any other simpler concepts?

This second problem is fundamental (see [11]). Now we define:

Simple concepts: Let X be any object of order 1. ^{0}X is a *simple concept.*

The suspicious feature of our analyses consisted in *automatically associating simple expressions with simple concepts* (see [18]). The respective assumption, viz. that the meaning of a simple expression is a simple concept, is false: In any stage of its development a language **L** contains many simple expressions that are *abbreviations* in virtue of to some system of *definitions.* A classical form of definitions ('equational definitions') is

$$\mathit{Definiendum} \ = \mathit{Definiens}$$

where Definiendum is a simple expression, and Definiens is a complex expression whose components have been already introduced into the language.

After **L** has been enriched by such a definition we must state that the simple expression playing the role of *Definiendum* has been endowed with the meaning of the complex expression called *Definiens.*

As an example we can use a). Let **L** contain the expressions *'factor of', 'the number of', 'to be', 'natural number',* the names of natural numbers, etc. The grammar makes it possible to create the complex expression *"natural numbers such that the number of their factors is 2."* Let us try to propose an analysis of this expression. Let us change our atomic types: instead of τ (real numbers) we choose ν (natural numbers). Now we proceed as follows:

Types: the N(umber of) / $(\nu\ (o\nu))$
...be a F(actor of) / $(o\nu\nu)$
$=$ (is) / $(o\nu\nu)$
2 / ν

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Synthesis:

(P) $\lambda x\ [^0{=}\ [^0\mathrm{N}\ \lambda y\ [^0\mathrm{F}\ y\ x]\ ^0 2]\ (x,\ y \rightarrow \nu)$

Now consider the definition:

(D) *Prime = a number such that the number of its factors is 2.*

A moment's reflection will confirm our claim that (P) is an analysis of the right-hand side of the identity (D). It means that from the viewpoint of TIL the meaning of *'Prime'* in the given language is (P) rather than ^{0}Prime.[21]

To function as a definition in the language **L**, (D) must satisfy the requirement that all expressions occurring on the right-hand side of (D) were already expressions of **L**. This means that the simple concepts $^0{=}$, ^{0}N, ^{0}F, ^{0}2 should be meanings of those expressions. But take, e.g., the simple concept ^{0}F. Do the users of **L** really understand the word *'factor'* because they can use the procedure that consists in taking the whole infinite set of ordered pairs of natural numbers where the first number is a factor of the second number? Or does the story repeat? Why should the concept connected with the expression *factor* be just the trivialization ^{0}F?

Thus, we see that a certain *relativization* is necessary. If the terminal node of a TTT which is associated with a simple expression E is occupied by a simple concept C then it will be correct only under the assumption that the meaning of that expression is a simple concept, which again depends on the conceptual system that underlies the given stage of development of **L**. Thus we have to define conceptual systems (see [19]).

Conceptual systems: Let $\alpha = \{\alpha_1,...,\alpha_m\}$ be a set of some atomic types, $\mathbf{C} = \{C_1,...,C_k\}$ a set of some simple concepts, where the types of the objects constructed by the members of $\mathbf{C}$ are types over α. A *conceptual system* $\mathbf{CS}_{\alpha,\mathbf{C}}$ is the union $\mathbf{C} \cup \mathbf{D}$, where $\mathbf{D}$ is the set $\{C_{k+1}, \ldots\}$ of those composed concepts (distinct from the members of $\mathbf{C}$) the simple subconcepts of which are at most members of $\mathbf{C}$. The members of α will be called *preconcepts* (of $\mathbf{CS}_{\alpha,\mathbf{C}}$), the members of $\mathbf{C}$ will be called *primitive concepts* (of $\mathbf{CS}_{\alpha,\mathbf{C}}$) and the members of $\mathbf{D}$ *derived concepts* (of $\mathbf{CS}_{\alpha,\mathbf{C}}$).

Comments: Preconcepts do not satisfy the definition of concepts, yet we need to consider them when analyzing expressions containing, e.g., the expression *'real number'*. For example, consider the sentence

There are real numbers which are irrational.

Our type-theoretical analysis will have
There are $(= \exists)\ /\ (o(o\tau))$
Ir(rational) $/\ (o\tau)$
Real numbers / ?

[21] The concept ^{0}Prime can be said to be at best *equivalent* to the concept (P).

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In a system where the type τ of real numbers is among the atomic types it is impossible to handle the object *real numbers* in another way than *via* variables, as follows,

$$[{}^0\exists\ \lambda x\ [{}^0\mathrm{Ir}\ x]],\ (x \to \tau)$$

or, equivalently, exploiting the type of the respective object (here: $(o\tau)$):

$$[{}^0\exists\ {}^0\mathrm{Ir}].$$

Thus we can analyze even the expressions that name the atomic types, like here *'real numbers', 'possible worlds', 'individuals', 'truth-values'.* Therefore, we make the atomic types play the role of *preconcepts.*

The idea of conceptual systems is that the users of a language in a stage of its development have at their disposal some simple procedures (which we called *primitive concepts*) which can be combined so that more and more complex procedures (concepts) arise. This idea is of importance for our present topics, as we will show just now.

Consider the sentence

Some bachelors are crazy.

Types: Some / $((o(o\iota))(o\iota))$
Bachelor / $(o\iota)_{\tau\omega}$
Crazy / $(o\iota)_{\tau\omega}$

Synthesis:

(B) $$\lambda w\lambda t\ [[{}^0\mathrm{Some}\ {}^0\mathrm{Bachelor}_{wt}]\ {}^0\mathrm{Crazy}_{wt}]$$

The logical form of our sentence according to (B) would be (tentatively):

$$\lambda w\lambda t\ [[\mathrm{X}\ \mathrm{Y}_{wt}]\ \mathrm{Z}_{wt}]$$

with $\mathrm{X}{:}((o(o\iota))(o\iota))$, $\mathrm{Y}{:}(o\iota)_{\tau\omega}$, $\mathrm{Z}{:}(o\iota)_{\tau\omega}$.

Whoever assents to the sentence above will, of course, be commited to the sentence *Some men are crazy.* The logical form of (B) does not, however, justify this inference. How come that a normal (wo)man is ready to come to that conclusion? A normal user of the respective language (say, English) knows that the simple expression *'bachelor'* has got its meaning *via* a definition. Let us simplify a little and suppose that the respective definiens is

an unmarried (elderly) man.

Now the types are
Unmarried / $((o\iota)_{\tau\omega}(o\iota)_{\tau\omega})$
(adjectival modifier: applied to a property it returns a (new) property)
Man / $(o\iota)_{\tau\omega}$.

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A new synthesis:

(B') $\lambda w \lambda t\ [[^0\text{Some}\ [^0\text{Unmarried}\ ^0\text{Man}]_{wt}\ ^0\text{Crazy}_{wt}]]$

so that the new logical form will be

$$\lambda w \lambda t\ [[\text{X}\ [\text{Y Z}]_{wt}\ \text{U}]]$$

with X: $((o(o\iota))(o\iota))$, Y: $((o\iota)_{\tau\omega}(o\iota)_{\tau\omega})$, Z: $(o\iota)_{\tau\omega}$, U: $(o\iota)_{\tau\omega}$,

which seems to make it possible to infer that some men are crazy, etc.[22]

In [19] the claim is defended that analyticity is relative to conceptual systems. Now we will examine this claim in connection with our explication of *logical form.* Let us compare two sentences:

(Bachelor) *All bachelors are men.*

(Unmarried) *All unmarried men are men.*

'To be analytic' for an expression means that its reference is given by its meaning. This concise formulation calls for an explanation: we have said already that meanings must be structured, *'algorithmically complex'.* Thus, we accepted the construction associated with an expression E according to the Parmenides Principle as the meaning of E. This meaning/concept constructs (in the better case) an object, which is called the *denotation of* E. In particular, if E is an empirical sentence, then the denotation of E is a *proposition.* The denotation (if any) of E is thus always determined by the meaning of E. The *reference* of an empirical expression E is the value (if any) of the denotation of E (i.e., of the intension that is this denotation) in the actual world at the present time. Thus, the reference of an empirical sentence is a truth-value. Logical analysis cannot reveal the reference of an empirical expression. No distinction between denotation and reference can be defined in the case of non-empirical expressions.

Therefore, saying that *to be analytic for an expression means that its reference is given by its meaning* and supposing that 'to be given' is meant as 'to be logically (unambiguously) determined', we claim that an analytic expression is not an empirical expression.

Our question is now: is the sentence (Bachelor) analytic(ally true)?

Let us suppose that the conceptual system at our disposal contains (among its primitive concepts) the following simple concepts (the types concern the constructed objects):

$^0\text{All} \rightarrow ((o(o\iota))\ (o\iota))$
$^0\text{Bachelor} \rightarrow (o\iota)_{\tau\omega}$
$^0\text{Man} \rightarrow (o\iota)_{\tau\omega}$.

[22] From the theory of modifiers it follows that this inference is possible, under some further assumptions; see below the characteristics of intersective modifiers

Our analysis of (Bachelor) is:

(Bach) $$\lambda w\lambda t\ [[^0\mathrm{All}\ ^0\mathrm{Bachelor}_{wt}]\ ^0\mathrm{Man}_{wt}],$$

the respective logical form is

$$\lambda w\lambda t[[\mathrm{X}\ \mathrm{Y}_{wt}]\ \mathrm{Z}]$$

with X: $((o(o\iota))(o\iota))$, Y: $(o\iota)_{\tau\omega}$, Z: $(o\iota)_{\tau\omega}$.

Does the meaning of the sentence, i.e., the construction (Bach), logically unambiguously determine its reference, i.e., **T**?

The problem is with ^{0}Bachelor. This is a simple concept of the property of being a bachelor. Thus on the one hand the property of being a man is inseparably connected with the property bachelor; on the other hand, there is nothing in the simple procedure that identifies bachelorship which would also identify the property of being a man. Thus it seems that this example (as so many similar ones) brings out the deep roots of the well-known distinction between analytically and logically true sentences. As we have often been told, an analytically true sentence can be in an appropriate way 'translated' so that the result is a logically true sentence. We want to show that this 'appropriate translation' can be explicated as a transition to another conceptual system whose primitive concepts make it possible to ontologically define objects that are constructed in the original conceptual system by simple concepts.[23] In our example it means that we analyze the sentence (Unmarried), employing a conceptual system that contains simple concepts

^{0}Unmarried $\rightarrow ((o\iota)_{\tau\omega}(o\iota)_{\tau\omega})$
^{0}Man $\rightarrow (o\iota)_{\tau\omega 1}$.

We get

$$\lambda \mathrm{w}\lambda \mathrm{t}\ [[^0\mathrm{All}\ [^0\mathrm{Unmarried}\ ^0\mathrm{Man}]_{\mathrm{wt}}]\ ^0\mathrm{Man}_{\mathrm{wt}}].$$

The logical form is now

$$\lambda w\lambda t\ [[\mathrm{X}\ [\mathrm{Y}\ \mathrm{Z}]_{\mathrm{wt}}]\ \mathrm{Z}_{\mathrm{wt}}]$$

with X: $((o(o\iota))(o\iota))$, Y: $((o\iota)_{\tau\omega}(o\iota)_{\tau\omega})$, Z: $(o\iota)_{\tau\omega}$.

It seems now that Evans' distinction between formal validity and validity in virtue of semantic structure (= structural validity[24]) can be supported by the preceding analysis. We could generalize the criterion given by formal validity and offer the following definition: A sentence S is formally valid iff the semantic contribution of its components[25] suffices to guarantee that S is true. In this sense we

[23] For ontological definition see [9, p. 12], or [17, Chapter 8.]
[24] See [26, p. 308]
[25] i.e., not only of logical constants

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can say that (Bachelor), as well as (Unmarried), is formally valid. The respective logical forms show, however, that only (Unmarried), and not (Bachelor), could be structurally valid, viz. if valid at all.

We now formulate sentences whose logical form is $\lambda w \lambda t$ $[[\mathrm{X}\ [\mathrm{YZ}]_{\mathrm{wt}}]\ \mathrm{Z}_{\mathrm{wt}}]$. For example

Some black cats are cats.

All flying horses are horses.

Unfortunately, natural languages are very complicated and even 'insidious'. Some counterexamples can be found even in our relatively simple case: the sentence

(C1) *All forged banknotes are banknotes.*

should be structurally valid according to the associated logical form, but it is analytically false. Another example:

(C2) *Some flying horses are horses.*

This sentence is empirically false, hence not structurally valid. Two options of the solution of these unpleasant cases are:

1) We should look for a modification of our tentative definition of logical form that would respect counterexamples such as (C1), (C2), i.e., these sentences should be associated with another logical form than the sentence (Unmarried).

2) We could insist on our claim that even (C1) and (C2) share the logical form $\lambda w \lambda t$ $[[\mathrm{X}\ [\mathrm{Y\ Z}]_{\mathrm{wt}}]\ \mathrm{Z}_{\mathrm{wt}}]$, but then, of course, the explication of logical form would not satisfy our expectation, viz. that logical form would enable us to detect such important semantic characteristics as validity and entailment.

If *logical form* has to parallel the structure of *meaning*, the second option is not viable. So we will examine the way offered by the first option.

First of all, we need to find out why the form $\lambda w \lambda t$ $[[\mathrm{X}\ [\mathrm{Y\ Z}]_{\mathrm{wt}}]\ \mathrm{Z}_{\mathrm{wt}}]$ failed to guarantee validity in the case of (C1). Actually, there is no logical reason why it should. Then we must ask why our intuition has it that if such adjectives like *'unmarried'*, *'black'*, and even *'flying'* are analyzed as modifiers corresponding to Y: $((o\iota)_{\tau\omega}(o\iota)_{\tau\omega})$ the validity is sometimes preserved.

As we might expect, the solution has to be sought in the semantics of modifiers. Since our semantics is procedural, and therefore essentially structured, we should not be surprised when it proves to influence the logical form. In our case we have to take into account that adjectives like *'unmarried'*, etc., denote *intersective* modifiers, for which the following rule holds:

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> Let M be an intersective modifier and A an expression denoting a property. Then:
> $MA(x)$ iff $M(x)$ and $A(x)$.

From the viewpoint of logical analysis this rule is not satisfactorily formulated: Let M be a modifier, type $((o\iota)_{\tau\omega}(o\iota)_{\tau\omega})$, and let M' denote a property, type $(o\iota)_{\tau\omega}$.[26] Semantically, M in $MA(x)$ denotes another object, viz. M, than M in $M(x)$, viz. M'.[27]

In particular, let Unmarried / $((o\iota)_{\tau\omega}(o\iota)_{\tau\omega})$ and Unmarried' / $(o\iota)_{\tau\omega}$. The construction

$$\lambda w \lambda t\ [{}^0\text{All}\ [{}^0\text{Unmarried}\ {}^0\text{Man}]_{wt}]\ {}^0\text{Man}_{wt}]$$

does not guarantee the validity of the sentence (Unmarried), unlike the construction

$$\lambda w \lambda t\ [{}^0\forall\ \lambda x\ [{}^0\supset\ [{}^0\wedge\ [{}^0\text{Unmarried'}_{wt}\ x][{}^0\text{Man}_{wt}\ x]][{}^0\text{Man}_{wt}\ x]]].$$

(Mind that All and $\forall$ are objects of different types: All/$((o(o\iota))(o\iota))$, $\forall/(o(o\iota))$.)

This last construction seems to contravene the Parmenides Principle: The sentence (Unmarried) does not contain names of implication or conjunction. A very natural way of solving this puzzle seems to consist in the following consideration:

If a concatenation of an adjective A and a substantive S has to be analyzed then the type associated with A depends on whether A denotes or not an intersective modifier. In the first case the concatenation AS is only an *abbreviation* encoding the intersection of the classes that are values of the properties associated with A' and S in the given world-time. In the second case the type associated with A is $((o\iota)_{\tau\omega}(o\iota)_{\tau\omega})$, which does not preserve validity, as we have already shown.

So we could derive the logical form as follows:

$$\lambda w \lambda t\ [\text{X}\ \lambda x\ [\text{Y}\ [\text{Z}\ [\text{U}_{wt}\ x][\text{V}_{wt}\ x]][\text{V}_{wt}\ x]]]$$

with $\quad$ X: $(o(o\iota))$, Y, Z: (ooo), U, V: $(o\iota)_{\tau\omega}$, $x \rightarrow \iota$.

Now we can show that our tentative definition of logical form has been wrong in that it does not fulfil the purpose of the explication. Indeed, to generate counterexamples is a trivial task. One simple instance: Let disjunction instead of conjunction be represented by Z.

This time the remedy is easy. Wherever *pure logical objects*, in particular logical connectives, quantifiers and identity are mentioned[28] in the given construction they *must be represented in the logical form*, i.e., they *must not be replaced with variables* of the respective type.

Thus the logical form of the sentence (Unmarried) will be

[26] In both cases we can generalize, i.e., any type α instead of ι would do.
[27] There is a semantic connection between M and M'. M' is definable in terms of M.
[28] via trivialization

$$\lambda w \lambda t\ [{}^0\forall\ \lambda x\ [{}^0\supset\ [{}^0\wedge\ [\mathrm{X}_{wt}\ x][\mathrm{Y}_{wt}\ x]][\mathrm{Y}_{wt}\ x]]]$$

with the obvious type-theoretical analyses.

Now what about (C2)?

The adjective *'flying'* denotes an intersective modifier, i.e., properly speaking a property. Thus the analysis of (C2) will be

$$\lambda w \lambda t\ [{}^0\exists\ \lambda x\ [{}^0\wedge\ [{}^0\wedge\ [{}^0\mathrm{Flying}_{wt}\ x]\ [{}^0\mathrm{Horse}_{wt}\ x]]\ [{}^0\mathrm{Horse}_{wt}\ x]]]$$

The logical form is

$$\lambda w \lambda t\ [{}^0\exists\ \lambda x\ [{}^0\wedge\ [{}^0\wedge\ [\mathrm{X}_{wt}\ x][\mathrm{Y}_{wt}\ x]][\mathrm{Y}_{wt}\ x]]]$$

(types obvious). Now $\exists$ is the class of the non-empty classes (here: classes of individuals). It is a logical truism that the intersection $A \cap B$ of two non-empty classes need not be non-empty. Thus the logical form underlying (C2) does not guarantee validity, which is as it should be.

Remark: It should be clear by now that the (most fine-grained) logical form of the sentence

a) *Some primes are even*

is not the same as of the sentence

b) *Some mammals are predators.*

The logical form of a) is (X: $(o\tau)$, Y: $(o\tau)$, $\mathrm{Some}_\tau/((o(o\tau))(o\tau))$):

$$[[{}^0\mathrm{Some}_\tau\ \mathrm{X}]\ \mathrm{Y}],$$

the logical form of b) is (X: $(o\iota)_{\tau\omega}$), Y: $(o\iota)_{\tau\omega}$, $\mathrm{Some}_\iota/((o(o\iota))(o\iota))$):

$$\lambda w\ \lambda t\ [{}^0\mathrm{Some}_\iota\ \mathrm{X}_{wt}]\ \mathrm{Y}_{wt}].$$

From the logical form of b) we can immediately see that the sentence is empirical (unlike a)), which is not an inferentially neutral fact.

Concluding we define ***logical form of an expression E with respect to a conceptual system CS:***

> Let E be an expression and let C_E be the construction that is the best analysis of E with respect to a conceptual system *CS*. The *logical form of E with respect to CS* is a hybrid expression that differs from the record of C_E just by replacing records of trivializations of extra-logical objects with linguistic variables as follows:
> Let ^{0}X be the given trivialization and let X / α for a type α. Let V be a variable replacing ^{0}X. Then the range of V is the collection of all members of α.

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6 Summary.

We have defended the following claims:

1. *The notion of logical form is a semantic notion.*
2. *One and the same expression of a natural language can be associated with various logical forms.*
3. *The more fine-grained the logical form of an expression E is, the more correct inferences based on E can be drawn.*
4. *To serve its purpose, an explication of the logical form of E must be based on the maximally fine-grained structured meaning of E.*
5. *Transparent Intensional Logic provides a highly fine-grained explication of meaning.*
6. *Which meaning has to be associated with an expression E is dependent on the conceptual background, i.e., on the conceptual system w.r.t. which the logical analysis of E is to be performed.*
7. *Given a conceptual system* $\boldsymbol{C}$, *any expression E can be associated with a particular analysis (construction, concept) that is the best one w.r.t.* $\boldsymbol{C}$.
8. *Every well-formed expression E of a given language can be associated with a logical form that is unambiguously derived from the best analysis w.r.t. a conceptual system* $\boldsymbol{C}$.

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